A recipe for black box functors

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From chemical reaction networks to tensor networks to finite state automata, network diagrams are often used to represent and reason about interconnected systems. What makes such a language convenient, however, is not just that these diagrams are intuitive to read and work with: it's that the notion of networking itself has meaning in the relevant semantics of the diagrams—the chemical or computational systems themselves.

More formally, recent work has used monoidal categories, and in particular *hypergraph* categories, to describe the algebraic structure of such systems, including electrical circuits, signal flow graphs, Markov processes, and automata, among many others. In this approach, diagrams are formalised as morphisms in a hypergraph category which represents the syntax of the language, and they are interpreted in another hypergraph category which models the semantics of the language. What matters then, is that this process of interpretation preserves the network operations: that this map forms what is known as a hypergraph functor.

In these hypergraph categories, the objects model interface or boundary types, and semantic interpretation often has the effect of hiding internal structure, and reducing the combinatorial, network-style diagram description of a system to the data that can be obtained via interaction, or composition, with other systems. In other words, semantic interpretation has the effect of wrapping the network in a 'black box'. We hence, informally, call a hypergraph functor that describes the semantics of a system a black box functor. This paper describes a general method for constructing such functors.

One method for defining the domains of these functors (thus, hypergraph categories) is given by the decorated cospans construction. The main theme of this paper is a careful study of a generalisation of this construction, known as *decorated corelations*. Decorated cospans constructs a hypergraph category from a finitely cocomplete category $\mathcal{C}$ and a lax symmetric monoidal functor $F\colon (\mathcal{C}, +) \to (\mathsf{Set}, \times)$. Although useful for recording the information present in open networks, decorated cospans sometimes fail to be efficient, since they can carry redundant information that is inaccessible from the boundary. Decorated corelations solves this issue by also requiring a factorisation system $(\mathcal{E},\mathcal{M})$ on $\mathcal{C}$, and extending F to a functor on a certain subcategory of $\mathsf{Cospan}(\mathcal{C})$.

Our main result is the functoriality of this construction. Indeed, we define a category $\mathsf{DecData}$ whose objects are *decorating data*: the tuples $(\mathcal{C},(\mathcal{E},\mathcal{M}),F)$ required for the decorated corelations construction. Write Hyp for the category whose objects are hypergraph categories and morphisms are hypergraph functors. Then, the decorated corelations construction defines a functor

$$(-)\mathsf{Corel}\colon \mathsf{DecData} \to \mathsf{Hyp}\,.$$

We prove this theorem by a characterisation, interesting in its own right, of the decorated corelations functor $(-)\mathsf{Corel}$ in terms of left Kan extension. To do this, we make use of a full subcategory of $\mathsf{DecData}$ that we call $\mathsf{CospanAlg}$, whose objects are finitely cocomplete categories $\mathcal{C}$ together with a lax symmetric monoidal functor $\mathsf{Cospan}(\mathcal{C}) \to \mathsf{Set}$.

Thus, we prove that the functor $(-)\mathsf{Corel}$ factors as the composite $\Phi\circ\mathsf{Kan}$, where these functors are part of adjunctions

$$\begin{tikzcd} \mathsf{Hyp} \arrow[r, shift left, "\mathsf{Alg}"] \arrow[r, phantom, "\perp"] & \mathsf{CospanAlg} \arrow[l, shift left, "\Phi"] \arrow[r, shift left, "\iota"] \arrow[r, phantom, "\top"] & \mathsf{DecData}\,. \arrow[l, shift left, "\mathsf{Kan}"] \end{tikzcd}$$

As a corollary of these investigations, we show that the unit of the adjunction $\mathsf{Alg} \dashv \Phi$ is a component-wise equivalence of hypergraph categories, which proves that every hypergraph category can be built, up to equivalence, from $\mathsf{DecData}$ via the decorated corelations construction. Thus, our category $\mathsf{DecData}$ contains all the necessary information for dealing with hypergraph categories.

Furthermore, the decorated corelations construction provides a strong set of tools to build black box functors between hypergraph categories obtained through the construction, but not between, say, a category obtained through the decorated corelations construction and one defined ad hoc. Making use of our results, we show that one can always (up to equivalence) work in the category $\mathsf{DecData}$, where clear tools for the creation of black box functors are provided.

References

[FS18] B. Fong and M. Sarazola, A recipe for black box functors `https://arxiv.org/abs/1812.03601`